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THE SPLITTANCE OF A GRAPH

by

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The splittance of an arbitrary graph is the minimum number of edges to be added or removed in order to produce a split graph (i.e. a graph whose vertex set can be partitioned into a clique and an independent set). The splittance is seen to depend only on the degree sequence of the graph, and an explicit formula for it is derived. This result allows to give a simple characterization of the degree sequences of split graphs. Worst cases for the splittance are determined for some classes of graphs (the class of all graphs, of all trees and of all planar graphs).

1. Introduction

A *split graph* is a graph whose set of vertices can be partitioned into a complete and an independent set. This class of graphs includes threshold graphs [3], an important subclass of matrogenic [5], as well as of matroidal [10] graphs etc. Split graphs have been characterized in [6] as those graphs G which have the property that both G and its complementary graph $\bar{G}$ are chordal, alternatively, split graphs are those graphs which do not have an induced subgraph isomorphic to $2K_2$ (two parallel edges), C_4 (a square) or C_5 (a pentagon). In the present paper, we introduce a measure of non-splitness (the "splittance") of an arbitrary graph. Of course, split graphs turn out to be precisely the graphs with zero splittance.

Some theoretical properties of the splittance are investigated and an explicit formula for it in terms of the degree sequence of the graph is derived. As a by-product, we obtain the result that, if a graph is split, then all graphs with the same degree sequence are split, and we get also a simple characterization of the degree sequence of split graphs. At the opposite end of the scale there are the maximally nonsplit graphs. For some broad classes of graphs (the class of all graphs, of all trees, and of all planar graphs) their members with maximum splittance are determined.

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2. Definitions and fundamental properties

Let $G=(V,E)\equiv(V(G),E(G))$ be a finite, loopless graph without multiple edges. If $S\subseteqq V$, we denote by ∂S the set of all edges having one endpoint in S and the other in $V-S$. A subset S of V is *complete* if the subgraph $G(S)$ induced by S is complete, and it is *independent* if $G(S)$ is a null graph (i.e. a graph without edges). A *clique* is a maximal complete subset.

A graph is split if its set of vertices can be partitioned into a complete and an independent set. Either set can by empty. Even if we impose the restriction that the complete set be a clique, the partition need not be unique. Given an arbitrary graph G, we define the *splittance* $\sigma(G)$ of G to be the minimum number of edges to be added to, or removed from G in order to obtain a split graph. The next Proposition indicates that the splittance is a reasonable measure of "non-splitness": in particular, property (ii) is consistent with the well-known fact that a graph is split if and only if its complementary graph is such [6].

Proposition 1.
(i) *G is split if and only if $\sigma(G)=0$.*
(ii) *For all graphs G, $\sigma(G)=\sigma(\bar{G})$.*
(iii) *If $G_1, \ldots, G_r$ are the connected components of G, then $\sigma(G)\geqq\sigma(G_1)+\ldots+\sigma(G_n)$.*

Proof. Properties (i) and (ii) are immediate. (iii) follows from the observation that any adding or removing of edges in G to obtain a split graph also makes $G_1, \ldots, G_r$ into split graphs. ▮

The main result of this section is Theorem 4, which gives an explicit formula for the splittance of a graph G in terms of the degree sequence of G. We need some preliminary lemmas.

For each $S\subseteqq V$, set

$$s(S)=\frac{1}{2}|S|(|S|-1)-|E(G(S))|+|E(G(V-S))|. \tag{1}$$

Actually, $s(S)$ is equal to (number of edges to be added to $G(S)$ in order to make $G(S)$ into a complete graph) plus (number of edges to be removed from $G(V-S)$ in order to make $G(V-S)$ into a null graph).

Lemma 2. $\sigma(G)=\min\limits_{S\subseteqq V} s(S)$.

Proof. Clearly $\sigma(G)\leqq s(S)$ for all $S\subseteqq V$. On the other hand, let $G'=(V,E')$ be a split graph, with complete set S^* and independent set $V-S^*$, obtained from G by the addition or the removal of a minimum number of edges. Because of the minimality assumption, no removed edge could have had an endpoint in S^* and no added edge could have had an endpoint in $V-S^*$. Hence $\sigma(G)=s(S^*)$. ▮

Let $d=(d_1, \ldots, d_n)$, with $d_1\geqq\ldots\geqq d_n$, be a graphic sequence, i.e. d is a sequence of non-negative integers for which there is some graph with n vertices $v_1, \ldots, v_n$

such that d_i is the degree of v_i. Define

(2) $$m(d) = \max\{k\colon\ 1 \leqq k \leqq n,\ d_k \geqq k-1\}$$

and, for $1 \leqq k < n$

(3) $$\sigma_k(d) = \frac{1}{2}\left\{k(k-1) - \sum_{i=1}^{k} d_i + \sum_{i=k+1}^{n} d_i\right\}.$$

When there is no danger of confusion, we shall simply write m and σ_k in place of $m(d)$ and $\sigma_k(d)$, respectively. The introduction of the index m and of the sequence σ_k is motivated by the next Theorem 4. The relevant properties of the sequence σ_k are summarized in the following lemma.

Lemma 3.

a) *$\sigma_k \geqq 0$ for $1 \leqq k < n$.*
b) *If $d_m > m-1$, the sequence $\{\sigma_k\}$ has a single minimum for $k=m$, it is strictly decreasing for $1 \leqq k < m$, and strictly increasing for $m \leqq k < n$.*
c) *If $d_m = m-1$, then $\{\sigma_k\}$ has two adjacent minima for $k=m-1$ and $k=m$, it is strictly decreasing for $1 \leqq k < m-1$ and strictly increasing for $m \leqq k < n$.*

Proof. a) is a consequence of the Erdős—Gallai [4] relations

(4) $$\sum_{i=1}^{k} d_i \leqq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \quad \text{for} \quad 1 \leqq k < n$$

which, together with "$\sum_{i=1}^{n} d_i$ even", are necessary and sufficient conditions for d to be graphic.

As for b), a direct computation shows that, for $1 < k < n$, one has $\sigma_k - \sigma_{k-1} = (k-1) - d_k$. This last expression is negative for $1 \leqq k \leqq m-1$ because, for any such k, $d_k \geqq d_m \geqq m-1 \geqq k-1$; and positive for $m+1 \leqq k < n$, by the definition of m.

Finally c) results from the remark that one has $\sigma_m < \sigma_{m-1}$ or $\sigma_m = \sigma_{m-1}$ according to whether $d_m > m-1$ or $d_m = m-1$. ∎

Theorem 4. *For any graph G,*

(5) $$\sigma(G) = \sigma_m(d)$$

where d is the degree sequence of G and m is defined as in (2).

Proof. For any $S \subseteq V$, one has

(6) $$\sum_{x\in S} d(x) = 2\left|E(G(S))\right| + |\partial S|$$

where $d(x)$ is the degree of the vertex x. From (1) and (6) it follows that

(7) $$s(S) = \frac{1}{2}\left\{|S|(|S|-1) - \sum_{x\in S} d(x) + \sum_{x\in V-S} d(x)\right\}.$$

Since the sequence d is non-increasing, by putting $k=|S|$, we have

$$\sum_{x\subseteq S} d(x) \leqq \sum_{i=1}^{k} d_i, \qquad \sum_{x\in V-S} d(x) \geqq \sum_{i=k+1}^{n} d_i;$$

since $|S|=k$, it follows that

$$s(S) \geqq \frac{1}{2}\left[|S|(|S|-1)-\sum_{i=1}^{k} d_i+\sum_{i=k+1}^{n} d_i\right] = \sigma_k(d). \tag{8}$$

On the other hand, we notice that, for each k, there is at least one set S (namely, the set $S_k\equiv\{1, ..., k\}$) for which (8) holds as an equality. In conclusion, we have, taking into account Lemmas 2 and 3,

$$\sigma(G) = \min_{S\subseteq V} s(S) = \min_{1\leqq k<n} \min_{|S|=k} s(S) = \min_{1\leqq k<n} \sigma_k(d) = \sigma_m(d). \quad \blacksquare$$

Theorem 1 yields an explicit, easily computable formula for the splittance of a graph. For example, we have:

Corollary 5. *If p_n is the path on n vertices, C_n the cycle on n vertices, $K_{p,q}$ the complete bipartite graph on p and q vertices, and $R_{h,n}$ any regular graph of degree h on n vertices, then:*

$$\sigma(P_n) = n-4 \quad \text{for} \quad n \geqq 5 \quad (\text{but } \sigma(P_k) = 0 \text{ for } k \leqq 4),$$

$$\sigma(C_n) = n-3,$$

$$\sigma(K_{p,q}) = \frac{v(v-1)}{2}, \quad \text{where} \quad v = \min\{p, q\},$$

$$\sigma(R_{h,n}) = \frac{h(n-h-1)}{2}.$$

The proof of Theorem 4 yields the following simple procedure for obtaining a split graph from a given graph $G=(V, E)$ with a minimum number of additions or removals of edges:

1) Determine the degree sequence d of G and index the vertices so that $d_1\geqq \dots \geqq d_n$.
2) Compute $m=m(d)$.
3) Add to E all edges $(i,j)\in E$ such that $i<j\leqq m$.
4) Remove from E all edges $(i,j)\in E$ such that $m<i<j$.

Remark. The availability of an easily computable formula for the splittance can be constrasted with the well-known NP-completeness of the corresponding problem for bipartite graphs: namely, to find the minimum number of edges to be removed (here addition of edges does not help) from a given graph in order to make it bipartite [7].

3. Degree sequences of split graphs

In view of Proposition 1 (i), an immediate consequence of Theorem 4 is the following

Theorem 6. *A graph G with degree sequence d is split if and only if*

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i \tag{9}$$

where $m=m(d)$ is given by (2).

Notice that, by definition of m, one has $d_i \leqq m-1$ for all $i>m$, and thus $\sum_{i=m+1}^{n} d_i = \sum_{i=m+1}^{n} \min\{m, d_i\}$. Hence Theorem 6 can be rephrased by saying that a *graph G is split if and only if its degree sequence satisfies the m-th Erdős—Gallai inequality* (4) *with strict equality* (in [8] it has been proved that a graph G is threshold if and only if its degree sequence satisfies the first m Erdős—Gallai inequalities with strict equality).

Corollary 7. *If a graph is split, then every graph with the same degree sequence is also split.*

The next theorem describes the relationship between two split graphs having the same degree sequence. Given a graph $G=(V, E)$ and four vertices a, b, c, d of G such that $(a, b)\in E$, $(c, d)\in E$, $(a, c)\notin E$ and $(b, d)\notin E$, the operation of adding the edges (a, c) and (b, d) and removing the edges (a, b) and (c, d) is called an (edge) *interchange.*

Theorem 8. *If G and G' are split graphs with the same degree sequence and on the same vertex set V, there is a bipartition $\{C, I\}$ of V such that*

(i) *in both G and G', C is complete and I is independent*

(ii) *G' can be obtained from G through a finite sequence of interchanges of edges in ∂C.*

Proof. By a well-known result of Ryser [11, page 68] any two graphs with the same degree sequence can be obtained from each other through a finite sequence of interchanges. Thus, it will be enough to prove the theorem in the case when G' is obtained from G by a single interchange. The general case follows by induction.

Since G is split, there is a bipartition of V into a complete set C and an independent set I. If the interchange removes (a, b), (c, d) and adds (a, c), (b, d), we may assume, without loss of generality, that a belongs to C. But then $c\in I$, and thus $d\in C$, which in its turn implies $b\in I$. Hence, both (a, b) and (c, d) belong to ∂C. After the interchange, both (a, c) and (b, d) belong to ∂C. It follows that C is complete and I is independent also in G'. ∎

If G is a split graph with degree sequence d, the parameter $m(d)$ itself has a simple interpretation. As customary, let us denote by $\omega(G)$ the maximum cardinality of a clique and by $\chi(G)$ the chromatic number.

Theorem 9. *If G is a split graph with the degree sequence d, then*
(i) $\omega(G) = m(d)$
(ii) $\chi(G) = m(d)$.

Proof. (i) One must have $\omega(G) \leqq m(d)$. For, if V is the set of vertices of G, let $S \subseteqq V$ be such that $|S| > m = m(d)$. If the vertices of G are numbered from 1 to n so that $d_1 \geqq \ldots \geqq d_n$, S must contain a vertex $k > m$. From the definition of m, one has $d_k \leqq m-1 < |S|-1$ and thus S cannot be a clique (this argument holds for arbitrary graphs). Now, consider the particular set of vertices $S_m = \{1, \ldots, m\}$. One has

$$0 = \left\{m(m-1) - \sum_{i=1}^{m} d_i + \sum_{i=m+1}^{n} d_i\right\} \qquad \text{from (9)}$$

$$= 2s(S_m) \qquad \text{from (7)}$$

$$= \{m(m-1) - |E(G(S_m))|\} + |E(G(V-S_m))| \quad \text{from (1);}$$

since the two brackets are non-negative, one has $E(G(S_m)) = m(m-1)$ and $E(G(V-S_m)) = 0$. Hence S_m is complete, $V-S_m$ is independent and (i) is proved. (ii) We need exactly m colors for the vertices in S_m. Each vertex k in $V-S_m$ is certainly non-adjacent to some vertex in S_m, otherwise (i) would be contradicted: hence k can be given one of the existing m colors. Since $V-S_m$ is independent, the thesis follows. ▮

Remark. Since every induced subgraph of a split graph is also split, Theorem 9 implies that *split graphs are perfect.*

4. Maximally non-split graphs

In the present section, we consider the following general problem: "Given a family G of graphs, find the maximum splittance of a graph in G having n vertices". Such maximum splittance is determined for the cases when G is the family of all graphs, of all trees and of all planar graphs.

Theorem 10. *The maximum splittance of a graph with n vertices is*

$$\left[\frac{1}{2}\left(\binom{n}{2} - \left[\frac{n^2}{4}\right]\right)\right],$$

where $[x]$ denotes the greatest integer $\leqq x$.

Proof. Let G be a graph with n vertices. We have (taking into account Lemma 2)

$$\sigma(G) = \min_{S \subseteqq V} s(S) = \min_{S \subseteqq V} s(V-S) =$$

$$= \min_{S \subseteqq V} \min\{s(S), s(V-S)\} \leqq \min_{S \subseteqq V} \frac{s(S) + s(V-S)}{2}.$$

In view of formula (1), the last expression is equal to

$$\min_{S\subseteq V} \frac{|S|(|S|-1)+(n-|S|)(n-|S|-1)}{4}$$

$$= \min_{0\leqq k\leqq n} \frac{n(n-1)-2k(n-k)}{4} = \min_{0\leqq k\leqq n} \frac{2(k-n/2)^2+n^2/2-n}{4}$$

$$= \frac{1}{2}\left(\binom{n}{2}-\left[\frac{n^2}{4}\right]\right)$$

since the minimum is attained when $k=v$, with $v=\left[\frac{n}{2}\right]$. Since $\sigma(G)$ is an integer, we must also have

$$\sigma(G) \leqq \left[\frac{1}{2}\left(\binom{n}{2}-\left[\frac{n^2}{4}\right]\right)\right].$$

This bound is tight. In fact, for every n, there exists a graph G_n on n vertices with degree sequence $(v, v, \ldots, v)$ if $n\not\equiv 3 \pmod 4$ and $(v, v, \ldots, v, v-1)$ if $n\equiv 3 \pmod 4$. The existence of such graphs is well known, and can be easily deduced from the Erdős—Gallai theorem mentioned in the proof of Lemma 3. According to Theorem 4 (see also Corollary 5), we get

$$\sigma(G_n) = \frac{v(n-v-1)}{2}, \quad n \not\equiv 3 \pmod 4$$

$$= \frac{v^2-1}{2}, \quad n \equiv 3 \pmod 4.$$

In all cases,

$$\sigma(G_n) = \left[\frac{1}{2}\left(\binom{n}{2}-\left[\frac{n^2}{4}\right]\right)\right]. \quad \blacksquare$$

Thus, roughly speaking, the maximum splittance of a graph with n vertices, is about $\frac{n^2}{8}$, and regular — or almost regular — graphs of degree about $\frac{n}{2}$ are among the extremal ones.

Theorem 11. *The maximum splittance of a tree with n vertices is $n-4$ and the path P_n is a tree with maximum splittance.*

Proof. Consider an arbitrary tree T with n vertices. We can assume that T is not a star (i.e. $K_{1,n-1}$), because a star is split. Therefore, T contains a path P_4 of length 3. If we delete all the $n-4$ edges of T which do not belong to P_4, we obtain a split graph. Hence $\sigma(T)\leqq n-4$. On the other hand, we know, from Corollary 5, that $\sigma(P_n)=n-4$. $\blacksquare$

The rest of this section will deal with the study of worst cases for the splittance of planar graphs. A non-increasing sequence d of non-negative integers with even sum is said to be *planar* if there is at least one planar graph with degree sequence

d; otherwise the sequence is called *non-planar*. As a consequence of the well-known Euler formula, a necessary condition for a graphic sequence $d_1 \geqq \ldots \geqq d_n$ to be planar is [2]:

$$\sum_{i=1}^{n} d_i \leqq 6(n-2), \quad (n \geqq 3). \tag{10}$$

A sequence d satisfying (10) is called an *Euler sequence*. A sequence $d_1 \geqq \ldots \ldots \geqq d_n$ is called *regular* if $d_1 = d_n$ and *almost regular* if $d_1 = d_n + 1$. In the sequel, we denote by $k^{(n)}$ the sequence $(\overbrace{k, k, \ldots, k}^{n \text{ times}})$, by $h^{(m)} k^{(n)}$ the sequence $(\overbrace{h, h, \ldots, h}^{m \text{ times}}, \overbrace{k, k, \ldots, k}^{n \text{ times}})$ and so on; for the sake of simplicity we shall omit the upper index when it is equal to 1. We shall make extensive use of a number of known results concerning the planarity of regular or almost regular sequences. These results are summarized in the following two lemmas.

Lemma 12. *Every regular, graphic Euler sequence is planar, except for* $4^{(7)}$ *and* $5^{(14)}$.

Proof. See [9].

Lemma 13. *Every almost regular, graphic Euler sequence is planar, except for* $5^{(10)}4$, $5^{(12)}4$, $65^{(12)}$ *and* $65^{(14)}$.

Proof. See [12].

Theorem 14. *If p_n is the maximum splittance of a planar graph with m vertices, then*

— *for $n \leqq 12$, p_n is equal to the maximum splittance of any graph with n vertices, as given by Theorem* 10

— *for* $13 \leqq n \leqq 14$, $p_n = \left[\frac{5}{2}(n-6)\right] - 1$

— *for* $15 \leqq n \leqq 22$, $p_n = \left[\frac{5}{2}(n-6)\right]$

— *for* $n \geqq 23$, $\quad p_n = 3n - 27$.

Proof. For every n, an obvious upper bound for p_n is given by the maximum splittance of an arbitrary graph with n vertices. As we shall show later, such an upper bound is attained for all $n \leqq 12$. Thus, from now on, we may assume that $n \geqq 13$. Let G be a planar graph with n vertices, let d be its degree sequence and let $m = m(d)$ be defined as in (2). We distinguish two cases:

a) $m \geqq 7$. In view of (5), we have

$$\sigma(G) = \frac{1}{2}\left\{m(m-1) - 2\sum_{i=1}^{m} d_i + \sum_{i=1}^{n} d_i\right\}.$$

Since G is planar, $\sum_{i=1}^{n} d_i \leqq 6n - 12$. Observing that $d_i \geqq m-1$ for $1 \leqq i \leqq m$, we then have

$$\sigma(G) \leqq \frac{1}{2}\{6n - 12 - m(m-1)\} \leqq 3n - 27, \quad m \geqq 7. \tag{11}$$

b) $m \leqq 6$. Formulas (2) and (5) give, for every n and $m = m(d)$,

$$\sigma(G) \leqq \frac{1}{2} \sum_{i=m+1}^{n} d_i \leqq \frac{1}{2}(n-m)(m-1). \tag{12}$$

Therefore, we have

$$\sigma(G) \leqq \left[\frac{5}{2}(n-6)\right], \quad m \leqq 6. \tag{13}$$

From (11) and (13) it follows that

$$p_n \leqq \max\left\{\left[\frac{5}{2}(n-6)\right], \ 3n-27\right\}.$$

Since $\left[\frac{5}{2}(n-6)\right] \leqq 3n-27$ if and only if $n \geqq 23$, we obtain

$$p_n \leqq \left[\frac{5}{2}(n-6)\right], \quad 13 \leqq n \leqq 22, \tag{14}$$

$$p_n \leqq 3n-27, \quad n \geqq 23. \tag{15}$$

For $n=13$, 14 the upper bound given by (14) can be tightened. Let us consider a planar graph G with 13 vertices and degree sequence d. If $m=m(d)$ is greater than 6, then $\sigma(G) \leqq 12$ by (11); if $m \leqq 5$, then $\sigma(G) \leqq 16$ by (12). Finally, if $m=6$, one must have $d_i \geqq 5$ for $1 \leqq i \leqq 6$ and $d_i \leqq 5$ for $7 \leqq i \leqq 13$. Since $\sum_{i=1}^{13} d_i$ is even, one must have either $d_1 \geqq 6$ or $d_{13} \leqq 4$. Let us assume that $d_{13} \leqq 4$ (the case when $d_1 \geqq 6$ is treated in a similar way). Since the sequence $(5^{(12)}, 4)$ is non-planar by Lemma 13, at least one of the inequalities

$$d_1 \geqq 5, \ldots, d_6 \geqq 5, \quad d_7 \leqq 5, \ldots, d_{12} \leqq 5, \quad d_{13} \leqq 4$$

— and in fact at least two of them because of the even sum condition — must be strict. But then formula (5) gives $\sigma(G) \leqq 16$. Thus $p_{13} \leqq 16$. A similar argument, based on the non-planarity of the sequence $5^{(14)}$ (see Lemma 12) shows that $p_{14} \leqq 19$.

We shall now show that the bounds derived above are actually attained. For $n \geqq 23$, consider any plane triangulation [2] (or maximal planar graph in the terminology of other authors) with n vertices and maximum vertex-degree 6. One such plane triangulation certainly exists for all $n \geqq 23$: see, for example, [12, Fig. 2]. For any such graph, one cannot have $d_7 \leqq 5$, for otherwise one would have $\sum_{i=1}^{n} d_i \leqq 36 + 5(n-6) < 6n-12$, contradicting the fact that any plane triangulation has exactly $3n-6$ edges [2, Chap. 9]. Hence $d_1 = d_2 = \ldots = d_7 = 6$, and the splittance is $\frac{1}{2}\{42-42+6n-12-42\} = 3n-27$. The other cases, i.e. $1 \leqq n \leqq 22$, are covered by Table 1. The planarity of each extremal sequence in Table 1 can be deduced from Lemma 12 or Lemma 13 according as to whether the sequence is regular or almost regular.

Table 1

n	p_n	Extremal sequences
1	0	0
2	0	(0, 0)
3	0	(0, 0, 0)
4	1	$2^{(4)}$
5	2	$2^{(5)}$
6	3	$3^{(6)}$
7	4	$(3^{(6)}, 2)$
8	6	$4^{(8)}$
9	8	$4^{(9)}$
10	10	$4^{(10)}$
11	12	$4^{(11)}$
12	15	$5^{(12)}$
13	16	$(5^{(10)}, 4^{(3)})$
14	19	$(5^{(12)}, 4^{(2)})$
15	22	$(5^{(14)}, 4)$
16	25	$5^{(16)}$
17	27	$(5^{(16)}, 4)$
18	30	$5^{(18)}$
19	32	$(5^{(18)}, 4)$
20	35	$5^{(20)}$
21	37	$(5^{(20)}, 4)$
22	40	$5^{(22)}$

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